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Dimension spectra of fractal measures from uniform partitions and correlation integrals

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Abstract. The spectra of generalized dimensions D_q and of local exponents $f(\alpha)$ for fractal measures are evaluated by using the uniform partitions to compute the free energy. The numerical results obtained from optimal algorithms are compared with the analytical results obtained from the free energy evaluated with dynamical partitions, in the case of IFS measures. It is proved that the spectra D_q obtained from correlation integrals and dynamical partitions are the same even for $q < 1$. The spectra obtained from the uniform partitions agree with the analytical result of dynamical partitions for any $q > 1$ and for $q < 1$ only if the support of the measure is not fractal or if the dynamical partitions are a subset of the uniform partitions. The spectra obtained from a numerical approximation of the correlation integrals provide the correct result for any value of q. The algorithms based on the uniform partitions are fast and can be used for real-time analysis of digitized images.

1. Introduction

The local properties of fractal measures are specified by the scaling exponents of spheres. The global properties are described by the correlation integrals, defined as averages of the measure of a sphere raised to the power q, whose scaling exponents $\tau(q) = (q-1)D_q$ define the dimension spectrum D_q [1]. A simple relation between local and global properties was established in [2]: the Hausdorff dimension $f(\alpha)$ of the set of points with scaling exponent α is the Legendre transform of $\tau(q)$. A class of fractal measures is defined by the iteration of linear contracting maps with statistical weights (IFS) [3]. This is a fairly general class since sequences of such measures converge to the measures defined by iterating nonlinear contracting maps [4]. By using a technique based on the Mellin transform of correlation integrals [5] we explicitly evaluate the spectrum $\tau(q)$ for the IFS measures with integer $q > 1$. By analytic continuation the results extends to $q < 1$ whenever the correlation integral exists. At the same values of $\tau(q)$ the free energy computed from dynamical partitions vanishes. For fractal measures, whose generating IFS are not given, the free energy for uniform partitions must be computed. For $q > 1$ the sequence produced by uniform partitions converges to the same $\tau(q)$ obtained with dynamical partitions [6, 7]; for $q < 1$ no convergence results are known. We have made a systematic numerical investigation of the dimension spectra provided by the free energy with uniform partitions. Algorithms with optimal computational complexity have been developed and applied to fractal measures generated by IFS. Good convergence is reached for any fractal measure when $q > 1$ and for measures with a Euclidean support (unit interval or square) when

$q < 1$ and the result is the same as for the dynamical partitions. For fractal measures, whose dynamical partitions are not a subset of the uniform partitions, the convergence for $q < 1$ is poor and $\tau(q)$ is different from the result obtained with dynamical partitions.

Algorithms to evaluate the correlation integrals, using a discretization based on uniform partitions were also developed; convergence was observed to the spectrum $\tau(q)$ of dynamical partitions for any value of q.

A generic digitized image has support on the unit square and the uniform partition algorithm allows one to obtain a spectrum in real time (starting from 1024×1024 pixels of one byte). The correlation algorithm in this case is not recommended because it is very slow.

2. Correlation integrals

We consider a system $M = (M_1, \ldots, M_s)$ of linear maps with statistical weights $(p_1, \ldots, p_s)$. Letting

$$M_i(\boldsymbol{x}) = \lambda_i R(\alpha_i)\boldsymbol{x} + \boldsymbol{b}_i \qquad 0 < \lambda_i < 1 \qquad \boldsymbol{x} \in I = [0, 1]^d \tag{2.1}$$

where R denotes an orthogonal matrix, we assume that there is a connected set $I_0 \subseteq I$ such that its images $I_j = M_j(I_0)$ are disjoint subsets of I_0. The iterations $M^n(A_0)$ converge, for any $A_0 \subseteq I_0$, to a fractal attractor $\mathcal{A}$. We define the dynamical partitions $\mathcal{I}^{(n)}$ of I_0 by the recurrence $\mathcal{I}^{(n)} = M(\mathcal{I}^{(n-1)})$ where $\mathcal{I}^{(0)} = I_0$ and the dynamical partitions of the attractor $\mathcal{A}$ by $\mathcal{A}^{(n)} = \mathcal{I}^{(n)} \cap \mathcal{A}$. Any partition is the union of s^n disjoint sets

$$\begin{aligned} \mathcal{I}^{(n)} &= \bigcup_{k_1,\ldots,k_n} I_{k_1,\ldots,k_n} \qquad I_{k_1,\ldots,k_n} = M_{k_1} \circ \cdots \circ M_{k_n}(I) \\ \mathcal{A}^{(n)} &= \bigcup_{k_1,\ldots,k_n} A_{k_1,\ldots,k_n} \qquad A_{k_1,\ldots,k_n} = I_{k_1,\ldots,k_n} \cap \mathcal{A}. \end{aligned} \tag{2.2}$$

A measure μ on $\mathcal{A}$ is defined by assigning the value it takes on the sets of any of its partitions

$$\mu(A_{k_1,\ldots,k_n}) = p_{k_1} \ldots p_{k_n}. \tag{2.3}$$

The invariance of the measure with respect to M follows from the definition and reads

$$\mu(M(A)) = \sum_{k=1}^{s} \mu(M_k(A)) = \mu(A) \qquad \mu(M_k(A)) = p_k \mu(A). \tag{2.4}$$

Letting $f(\boldsymbol{x})$ be any function almost-everywhere continuous we define its average with respect to the measure μ by

$$\int f(\boldsymbol{x})\, \mathrm{d}\mu\,(\boldsymbol{x}) = \lim_{n\to\infty} \sum_{k_1,\ldots,k_n} f(\boldsymbol{x}_{k_1,\ldots,k_n}) \mu(A_{k_1,\ldots,k_n}) \tag{2.5}$$

where $\boldsymbol{x}_{k_1,\ldots,k_n}$ is any point in $A_{k_1,\ldots,k_n}$ (notice that the diameter δ tends to 0 as $n \to \infty$). From the invariance of the measure it follows that

$$\int f(\boldsymbol{x})\, \mathrm{d}\mu\,(\boldsymbol{x}) = \sum_{k=1}^{s} p_k \int f(M_k(\boldsymbol{x}))\, \mathrm{d}\mu\,(\boldsymbol{x}). \tag{2.6}$$

For any point $\boldsymbol{x} \in \mathcal{A}$ we consider the scaling exponent $\alpha(\boldsymbol{x})$ defined by the limit

$$\alpha(\boldsymbol{x}) = \lim_{r\to 0} \frac{\log \mu(\mathcal{A} \cap S(\boldsymbol{x}, r))}{\log r} \tag{2.7}$$

where $S(\boldsymbol{x}, r)$ is the sphere of centre $\boldsymbol{x}$ and radius r. If the limit does not exist the sup and inf limits define two distinct exponents $\overline{\alpha}(\boldsymbol{x}), \underline{\alpha}(\boldsymbol{x})$. The average of $\mu^{q-1}(\mathcal{A} \cap S(\boldsymbol{x}, r))$ defines the correlation integrals† whose scaling exponents are denoted by $\tau(q)$

$$C(q; r) = \int \mu^{q-1}(\mathcal{A} \cap S(\boldsymbol{x}, r))\, \mathrm{d}\mu\,(\boldsymbol{x}) \qquad \tau_C(q) = \lim_{r \to 0} \frac{\log C(q; r)}{\log r}. \tag{2.8}$$

For exactly self-similar fractals having equal scales and weights, the local scaling exponents are the same and their value is the Hausdorff dimension $D_H = -\log s / \log \lambda$. In this case $\tau_C(q) = (q-1) D_H$ is linear. For a generic attractor $\tau(q)$ is concave and defines a spectrum of dimensions

$$D_q = \frac{\tau(q)}{q-1}. \tag{2.9}$$

The result of the following holds.

Theorem. *For $q > 1$ the integral converges and the scaling exponent $\tau(q)$ of the correlation integral is the unique real solution of*

$$\sum_{j=1}^{s} p_j^q \lambda_j^{-\tau} = 1. \tag{2.10}$$

If the correlation integral converges for $q < 1$ then $\tau(q)$ is still the solution of (2.10).

The proof is based on the use of the Mellin transform

$$\Phi(\zeta; q) = \int_0^1 r^{-\zeta} \frac{\partial C}{\partial r}\, \mathrm{d}r \tag{2.11}$$

known as the energy integral. If C scales as r^τ then $\Phi \sim (\tau - \zeta)^{-1}$ has a pole at $\zeta = \tau$.

We consider first $q > 1$ integer so that $C(q, r)$ can be written as

$$C(r; q) = \int \prod_{i=1}^{q-1} \vartheta(r - \|\boldsymbol{x} - \boldsymbol{y}_i\|)\, \mathrm{d}\mu\,(\boldsymbol{y}_i)\, \mathrm{d}\mu\,(\boldsymbol{x}). \tag{2.12}$$

The Mellin transform takes the following form:

$$\Phi(\zeta; q) = \sum_{k=1}^{q-1} \int \|\boldsymbol{x} - \boldsymbol{y}_k\|^{-\zeta} \prod_{i \neq k}^{q-1} \vartheta(\|\boldsymbol{x} - \boldsymbol{y}_k\| - \|\boldsymbol{x} - \boldsymbol{y}_i\|) \prod_{j=1}^{q-1} \mathrm{d}\mu\,(\boldsymbol{y}_j)\, \mathrm{d}\mu\,(\boldsymbol{x}) \tag{2.13}$$

and the balance property of the measure allows one to write

$$\begin{aligned}\Phi(\zeta; q) = \sum_{j_0, j_1, \ldots, j_{q-1}} & p_{j_0} p_{j_1} \ldots p_{j_{q-1}} \int \|M_{j_0}(\boldsymbol{x}) - M_{j_k}(\boldsymbol{y}_k)\|^{-\zeta} \\ & \times \prod_{i \neq k}^{q-1} \vartheta(\|M_{j_0}(\boldsymbol{x}) - M_{j_k}(\boldsymbol{y}_k)\| - \|M_{j_0}(\boldsymbol{x}) - M_{j_i}(\boldsymbol{y}_i)\|) \prod_{j=1}^{q-1} \mathrm{d}\mu\,(\boldsymbol{y}_j)\, \mathrm{d}\mu\,(\boldsymbol{x}).\end{aligned} \tag{2.14}$$

Only when the indices $j_0, j_1, \ldots, j_{q-1}$ are equal can the factor $\|\cdots\|^{-\zeta}$ vanish and the contribution be proportional to $\Phi(\zeta, q)$ itself. The remaining terms define an entire function of ζ:

$$\Phi(\zeta, q) = \sum_{j=1}^{q} p_j^q \lambda_j^{-\zeta} \Phi(\zeta; q) + E(\zeta; q) \tag{2.15}$$

† In every definition where the measure of a set is raised to a power, it is implicitly assumed that the result is zero when the measure is zero, even if the power is negative: $\mu^q(B) \equiv 0$ if $\mu(B) = 0$ for any $q \in \mathbb{R}$.

and the result is proved. To extend the result to any real q is immediate since $\Phi(\zeta, q)$ is a meromorphic function in q and its value on the sequence of positive integer values of q allows one to continue it on the whole complex plane excluding the poles. On the other hand, the energy integral converges for any real $q > 1$ when ζ is real negative and the result is proved. For $q < 1$ the same conclusion holds provided that the correlation integral converges.

3. Free energies and uniform partitions

The dimension spectra are also introduced in the framework of the thermodynamic formalism. The free energy for the dynamical partitions is defined by

$$\mathcal{F}_D(q,\tau) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{k_1,\dots,k_n} \frac{\mu^q(A_{k_1,\dots,k_n})}{\delta^\tau(A_{k_1,\dots,k_n})} \tag{3.1}$$

where $\delta(A)$ denotes the diameter of the set A. Inserting the expression for the measures and the diameters of an IFS attractor with weights p_i and scales λ_i we obtain

$$\mathcal{F}_D(q,\tau) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{k_1,\dots,k_n} \frac{p_{k_1}^q \cdots p_{k_n}^q}{\lambda_{k_1}^\tau \cdots \lambda_{k_n}^\tau} = \log \sum_{j=1}^{s} p_j^q \lambda_j^{-\tau}. \tag{3.2}$$

The dynamical free energy $\mathcal{F}_D$ vanishes at the scaling exponents $\tau(q)$ of the correlation integrals. Moreover, $D_0 = -\tau(0)$ is the Hausdorff dimension. Indeed, the Hausdorff measure is defined on coverings $\mathcal{B}_\epsilon$ with sets B_i of diameter $\epsilon_i \leqslant \epsilon$ according to

$$H(\epsilon,\beta) = \inf_{\mathcal{B}_\epsilon} \sum_i \epsilon_i^\beta \tag{3.3}$$

and the limit for $\epsilon \to 0$ defines a function $H(\beta)$ which diverges for $\beta < D_H$ and vanishes for $\beta > D_H$, where D_H is the Hausdorff dimension. The dynamical partition $\mathcal{A}^{(n)}$ is the covering belonging to $\mathcal{B}_\epsilon$ with $\epsilon = \lambda^n$ (where λ is the largest of the scales λ_i), for which the minimum is achieved. As a consequence, the Hausdorff dimension satisfies $\lambda_1^{D_H} + \cdots + \lambda_s^{D_H} = 1$ which implies $\mathcal{F}_D(0, -D_H) = 0$.

The uniform partitions $\mathcal{U}^{(n)}$ correspond to a tesselation of the unit cube into cubelets of side 2^{-n}. Replacing, in the definition of the free energy, the elements $A_{k_1,\dots,k_n}$ of the dynamic partitions with the cubelets $c_i^{(n)}$ for $1 \leqslant i \leqslant 2^{nd}$ we obtain the free energy $\mathcal{F}_U$ which reads

$$\mathcal{F}_U = \tau + \lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{i=1}^{2^{nd}} \mu^q(\mathcal{A} \cap c_i^{(n)}). \tag{3.4}$$

The dimension spectrum of uniform partitions is defined by $\tau_U(q)/(q-1)$, where

$$\tau_U(q) = -\lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{i=1}^{2^{nd}} \mu^q(\mathcal{A} \cap c_i^{(n)}). \tag{3.5}$$

We consider the Legendre transform of $\tau(q)$, which enhances the deviations of $\tau(q)$ from a linear behaviour. Since $\tau(q)$ is concave, $\mathrm{d}^2\tau/\mathrm{d}q^2 < 0$, and its Legendre transform

$$f(\alpha) = \min_q (q\alpha - \tau(q)) \qquad \longrightarrow \qquad \alpha = \frac{\mathrm{d}\tau}{\mathrm{d}q} \tag{3.6}$$

exists and is also concave. The following properties are an immediate consequence of the definition:

$$\max f(\alpha) = D_0 \equiv D_H \qquad f(\alpha) = \alpha \qquad \text{for} \quad \alpha = D_1. \tag{3.7}$$

The interval $q < 1$ is mapped into $]\alpha_1, \alpha_{-\infty}]$, the interval $q > 1$ into $[\alpha_{+\infty}, \alpha_1[$ where $\alpha_1 = D_1$ with $f'(\alpha_1) = 1$ and $f(\alpha_{\pm\infty}) = D(\pm\infty)$. Since $q = \mathrm{d}f/\mathrm{d}\alpha$ the tangent is vertical at $\alpha_{\pm\infty}$ and the maximum value of $f(\alpha)$ is at $\alpha_0 > \alpha_1$, where $f(\alpha_0) = D_H$. It has been proved that

$$f(\alpha) = D_H(\mathcal{A}_\alpha) \qquad \mathcal{A}_a = \{\boldsymbol{x} : \alpha(\boldsymbol{x}) = a\}. \tag{3.8}$$

4. Numerical results

Algorithms of optimal computational complexity have been developed and implemented in order to evaluate the dimension spectra provided by the uniform partitions and to test the convergence, see the appendix. We have considered fractal measures on the unit interval discretized into a vector of length N^2 and fractal measures on the unit square discretized by an $N \times N$ matrix where the integer entries g_i are the grey tones, ranging in the interval $[0, N_g]$. The discretized measures on the unit interval and on the unit square correspond to partitions of order $2n$ and n, respectively, since $N = 2^n$.

Choosing $N = 2^{10}$ and $N_g = 2^{31} - 1$ (the bit corresponding to the sign being ignored in the four-byte integer assigned to g_i) the total memory storage required is 4 Mbytes and the highest-order partition is $n = 20$ and $n = 10$ for $d = 1, 2$, respectively.

The grey tone g_i was defined as the closest integer to $N_g \mu(\mathcal{A} \cap c_i^{(n)})$. The exact measure $\mu(\mathcal{A} \cap c_i^{(n)})$ is replaced with $g_i^{(n)}/G$, where $G = \sum g_i^{(n)}$ is very close to N_g.

In order to avoid errors introduced by the discretization process the number of grey tones must be sufficiently large. For a measure with equal scales λ, a rough estimate is given by $N_g \geqslant 2^{n \log p_1/\log\lambda}$ where p_1 is the smallest weight. If the measure is balanced the condition becomes $N_g \geqslant 2^{nD_H}$ and is satisfied with the above choice since $n = 20$, $D_H \leqslant 1$ for measures on the unit interval and $n = 10$, $D_H \leqslant 2$ for measures on the unit square. Indeed, the lowest-order m of a dynamical partition whose elements have a diameter less than or equal to 2^{-n} is $m = -n/\log_2\lambda$. Since the smallest measure of any element of the order-m dynamical partition is p_1^m from $N_g p_1^m \geqslant 1$, the above estimate follows.

For the lower-order partition $\mathcal{U}^{(n-1)}$ we replace the measure $\mu(c_i^{(n-1)})$ with $g_i^{(n-1)}/G$ where $g_i^{(n-1)}$ is computed by summing the grey tones $g_{i'}$ of the (two or four) cells $c_{i'}^{(n)}$ whose union is $c_i^{(n-1)}$. By iterating the process, the grey tones for all the partitions from order n to order zero are obtained. The total storage required is less than $2N^2$ integers and the total number of operations involved is less than $4N^2$ for fractal measures on the line and the plane respectively, see the appendix.

For the chosen value of n we consider the sequence

$$\tau_m^{(n)}(q) = -\frac{1}{m}\log_2 \sum \left(\frac{g_i^{(m)}}{G}\right)^q \qquad 1 \leqslant m \leqslant n. \tag{4.1}$$

The exact spectrum $\tau(q)$ is given by the limit of $\tau_n^{(n)}(q)$ as $n \to \infty$ or by the double limit for $n \to \infty$ and $m \to \infty$.

Since n is bounded to a finite value (10 or 20) we consider the sequence

$$\sum_i \left(\frac{g_i^{(m)}}{G}\right)^q = 2^{-m\tau(q)} F(m) \qquad 1 \leqslant m \leqslant n \tag{4.2}$$

where $F(m)$ denotes the prefactor of the scaling law. Since for $q > 1$ the sum approximates the correlation integral, we can determine the nature of the prefactors by looking at their Mellin transform. For a fractal measure with equal scales the singularities of the energy integral are the poles

$$\lambda^{-\zeta}\sum_{j=1}^{s} p_j^q = 1 = \mathrm{e}^{2\pi \mathrm{i}k} \qquad \zeta = \tau(q) - \mathrm{i}k\frac{\omega}{\log 2} \tag{4.3}$$

equally spaced on a line parallel to the imaginary axis, where

$$\tau(q) = \frac{\log(p_1^q + \cdots + p_s^q)}{\log\lambda} \qquad \omega = 2\pi\frac{\log 2}{\log\lambda}. \tag{4.4}$$

Taking the inverse Mellin transform the scaling behaviour of the correlation integral is given by $2^{-m\tau(q)}$ times a real periodic function of m of period ω, so that we can write

$$\sum_i (g_i^{(m)})^q \simeq G^q C_q(2^{-m}) = G^q 2^{-m\tau(q)} 2^{A_0+f(m)} \qquad f(m) = \sum_{k\geqslant 1} A_k \cos(mk\omega + \alpha_k). \tag{4.5}$$

It follows that a least-squares fit interpolation of the function

$$W^{(n)}(m;q) = \log_2 \sum_i (g_i^{(m)})^q = -m\tau(q) + q\log_2 G + A_0 + f(m) \tag{4.6}$$

should provide a good approximation to $\tau(q)$. For a generic measure the function $f(m)$ is quasi-periodic with decaying amplitudes, since the poles of the Mellin transform are no longer on a straight line, but a good approximation from the least-squares fit interpolation may still be expected.

A systematic investigation was made of IFS measures on the unit interval. Three distinct classes were considered:

(i) Measures whose dynamical partitions $\mathcal{I}^{(n)}$ are a subset of the uniform partitions $\mathcal{U}^{(n')}$ for some $n' \geqslant n$. Among them we distinguish the measures with support on [0, 1], and measures with fractal support such as those generated by

$$\begin{array}{ll} \text{(a) } M_1(x) = \tfrac{1}{2}x & M_2(x) = \tfrac{1}{2}x + \tfrac{1}{2} \\ \text{(b) } M_1(x) = \tfrac{1}{4}x & M_2(x) = \tfrac{1}{4}x + \tfrac{1}{2} \end{array} \tag{4.7}$$

for arbitrary weights.

(ii) Measures with support on [0, 1], whose dynamical partitions are not a subset of uniform partitions ($\lambda \neq 2^{-m}$ for equal scales), such as those generated by

$$\text{(c) } M_1(x) = \tfrac{1}{3}x \qquad M_2(x) = \tfrac{1}{3}x + \tfrac{1}{3} \qquad M_3(x) = \tfrac{1}{3}x + \tfrac{2}{3}. \tag{4.8}$$

(iii) Measures with fractal support, with dynamical partitions which are not a subset of uniform partitions, such as those on the ternary Cantor set generated by

$$\text{(d) } M_1(x) = \tfrac{1}{3}x \qquad M_2(x) = \tfrac{1}{3}x + \tfrac{2}{3}. \tag{4.9}$$

In figure 1 we show the spectrum $f(\alpha)$ for the measures generated by the maps defined above. For the measures (a), (b) and (c) the function $W^{(n)}(m;q)$ defined by (4.6) is linear in m (up to small oscillations) for any value of q; its slope $\tau(q)$ and the corresponding Legendre tranform $f(\alpha)$ are very close to the exact value.

For the measure (d) a nice linear behaviour is only observed for $q > 1$, but for $q < 1$ the deviations from linearity become significant and the slope $\tau(q)$, determined by a poor least-squares fit, differs from the theoretical values of $\tau(q)$ given by (2.10).

Figure 2 shows $W^{(n)}(m;q)$ versus $n-m$ for a set of q-values. For the measure (a) the linear behaviour holds for all values of q, whereas for the measure on the Cantor set the deviation from linearity for $q < 1$ is evident. For the Cantor set the extrapolated values for $\tau(q)$ differ very significantly from the exact result and the same discrepancy holds for the generalized dimensions D_q as shown by figure 3. It is not surprising that the Legendre transform for $\alpha > D_1$, is also very distant from the exact result. Notice that the computation of the Legendre transform is not a problem since the extrapolated value $\tau(q)$ is also still concave for $q < 1$.

A similar analysis has been carried out for measures on the unit square [0, 1]. We have considered a measure (a) generated by four maps which transform [0, 1] into four disjoint squares of side $\frac{1}{2}$ whose union is I, a measure (b) with fractal support generated by the first three of the previous maps, a measure (c) with support on [0, 1] generated by nine maps which map [0, 1] on disjoint squares of side $\frac{1}{3}$ whose union is the unit square and a measure (d) with support on the Serpinsky set. The function $W^{(n)}(m;q)$ is linear in m in the whole range

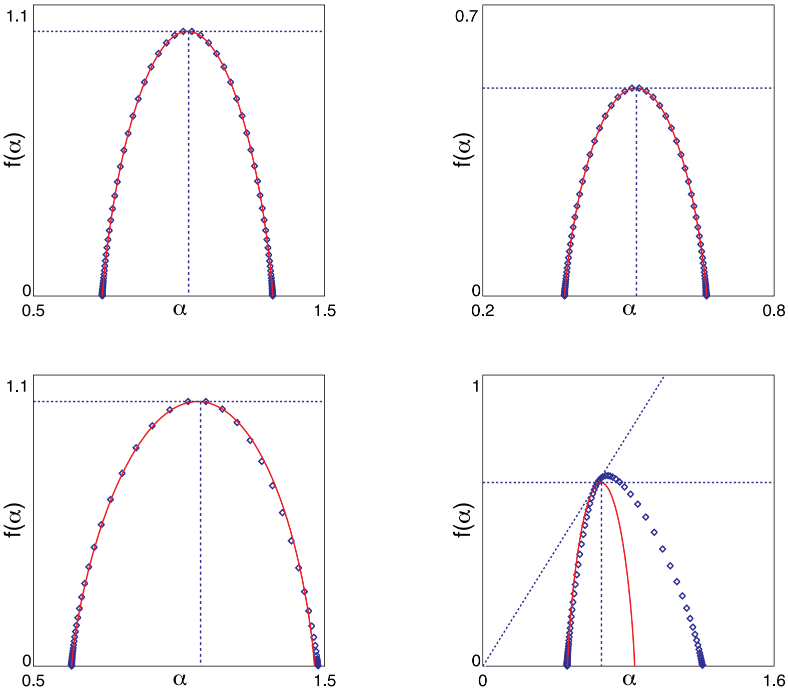

Figure 1. Legendre transform $f(\alpha)$ of the function $\tau(q)$ computed from uniform partitions (dots) and from dynamical partitions (continuous curves) for the measures generated by the maps (a) and (b) with weights $p_1 = 0.4$, $p_2 = 0.6$, by the map (c) with weights $p_1 = 0.2$, $p_2 = 0.5$, $p_3 = 0.3$ and by the map (d) with weights $p_1 = 0.4$, $p_2 = 0.6$. The diagonal line is $y = \alpha$ and is a tangent to the curve at $\alpha = \alpha_1$, where $f(\alpha_1) = \alpha_1 = D_1$.

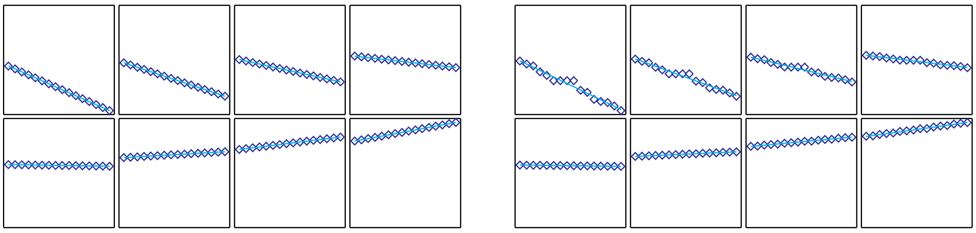

Figure 2. Plot of the function $W^{(n)}(m; q)$ versus $n - m$ for $q = -20, -15, -10, -5, 0, 5, 10, 15$ corresponding to the measure generated by the maps (a) with weights $p_1 = 0.4$, $p_2 = 0.6$ (left block) and to the measure on the ternary Cantor set with the same weights (right block). The values of q in each block increase from left to right, the upper row corresponding to the negative values of q and the lower row to the positive values.

$[1, n]$ and the slope determined by a least-squares fit agrees with the result (2.10) of dynamical partitions except for measure (d) when $q < 1$. This is evident in figure 4 where the Legendre transforms are compared; for the measure (d) the disagreement for $\alpha > D_1$ is evident. For the measures (a), (b) and (c) on the line and the plane no appreciable degradation of the results is observed by lowering the grey tones from 2^{31} to 2^{15}. The method has been applied to a variety of images not generated by IFS, and with support on the unit square. The linearity of $W(m, q)$ is usually good for any q and the Legendre transform is a concave function reaching the maximum value $D_H = 2$ even with a number of grey tones 2^8, which is typical of a digitized image.

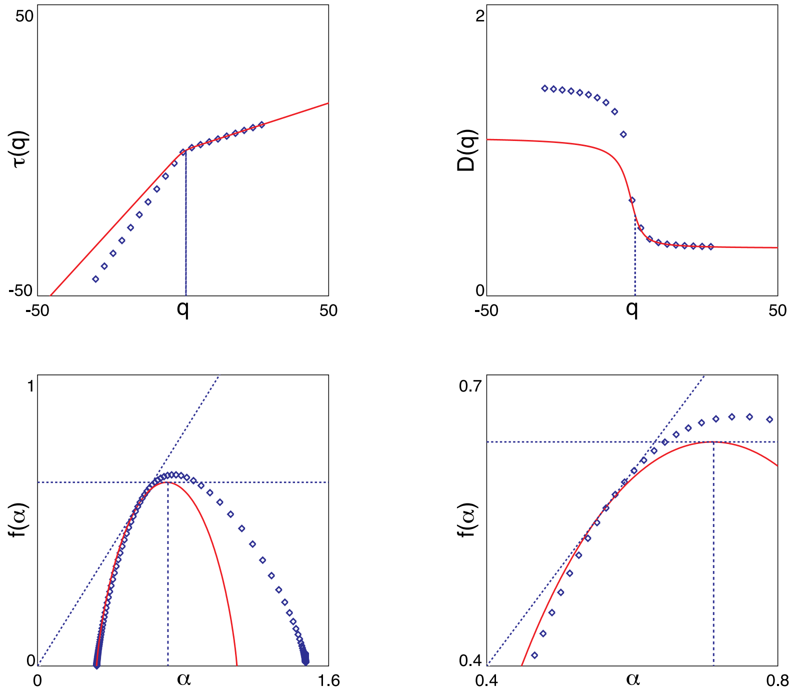

Figure 3. Comparison of the exponent $\tau(q)$ (upper left), dimension $D(q)$ (upper right), Legendre transform $f(\alpha)$ (lower left) and its enlargement (lower right) computed from uniform partitions (dots) and dynamical partitions (continuous lines and curves) for the measure on a ternary Cantor set with weights $p_1 = 0.3$, $p_2 = 0.7$.

Since in the generic case the free energy computed from uniform partitions does not provide the correct spectrum $\tau(q)$, one may wonder whether the correct result is numerically accessible. The answer is positive if one uses a correct discretization of the correlation integrals. In this case the average of a function is approximated by $\sum_i f(\boldsymbol{x}_i)g_i/G$ where $\boldsymbol{x}_i$ is any point in the cell $c_i^{(n)}$ of an order-n uniform partition. Convergence to the exact mean occurs for $n, N_g \to \infty$. The correlation integrals are computed for boxes of side $r = 2^{-n}(2^m + 1)$ centred on the cell $c_i^{(n)}$, where $n/2 \leqslant m \leqslant n$. Each box centred on the cell $c_i^{(n)}$, see figure 5, is the union of cells $c_j^{(n)}$ where the index j varies in some set $J(i, m)$ and we have

$$C(r, q) = \lim_{n\to\infty} \sum_i \mu(\mathcal{A} \cap c_i^{(n)}) \Bigg(\sum_{j\in J(i,m)} \mu(\mathcal{A} \cap c_j^{(n)}) \Bigg)^{q-1} . \tag{4.10}$$

The functions

$$W_C^{(n)}(m, q) = \log_2 \sum_i g_i^{(n)} \Bigg(\sum_{j\in J(i,m)} g_j^{(n)} \Bigg)^{q-1} \tag{4.11}$$

approximate $\log_2(G^q C(r, q))$ and consequently,

$$W_C^{(n)}(m, q) = (m - n)\tau(q) + q \log_2 G + A_0 + f(n - m) \tag{4.12}$$

where the last term comes from the corrections to the scaling law and we have neglected 2^{-m} compared with 1. The functions $W_C^{(n)}(m, q)$ prove to be linear in m for all the measures listed above including (d) when $q < 1$ and the slope determined by least-squares fit is in good

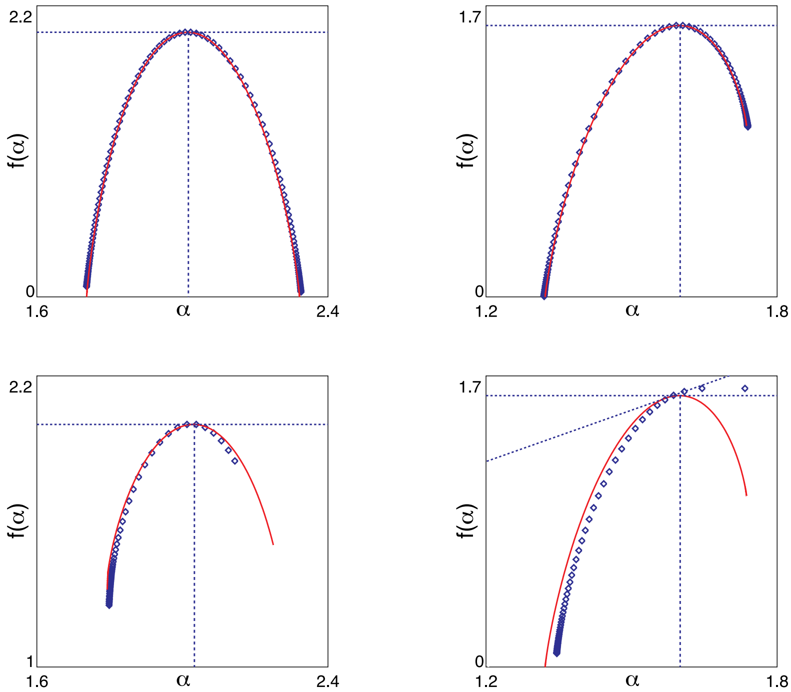

Figure 4. Legendre tranform $f(\alpha)$ of the function $\tau(q)$ computed from uniform partitions (dots) and from dynamical partitions (continuous curves) for the measures in the unit square generated by the maps (a), (b), (c) and (d) (from left to right) described in the text.

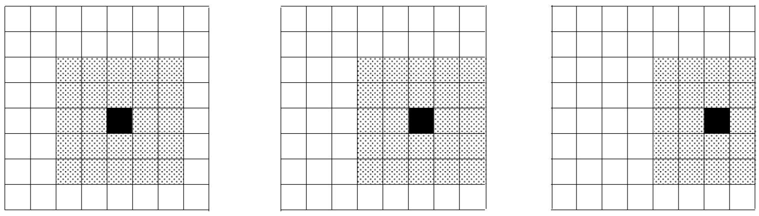

Figure 5. The boxes used to compute the correlation integrals are shown for a partition of order $n = 3$ of the unit square. The side of the boxes $r = \frac{5}{8}$ corresponds to $m = 2$ in the general expression $r = 2^{-n}(2^m + 1)$. The cell $c_i^{(n)}$ is black and the box centred on it is grey.

agreement with (2.10). In figure 6 we show the corresponding Legendre transform. It should be noted that when the support of the measure is the whole unit interval or square the algorithm becomes very slow. For this reason it is not recommended for the analysis of a digitized image, where fast computations are needed.

5. Conclusions

The algorithm introduced to compute the dimension spectra for uniform partitions is proposed for analysis of the local scaling properties of a digitized image. It has been shown that the spectra of fractal measures with support on the unit square are the same as the spectra provided by the dynamical partitions, within the numerical uncertainty, even for $q < 1$. As

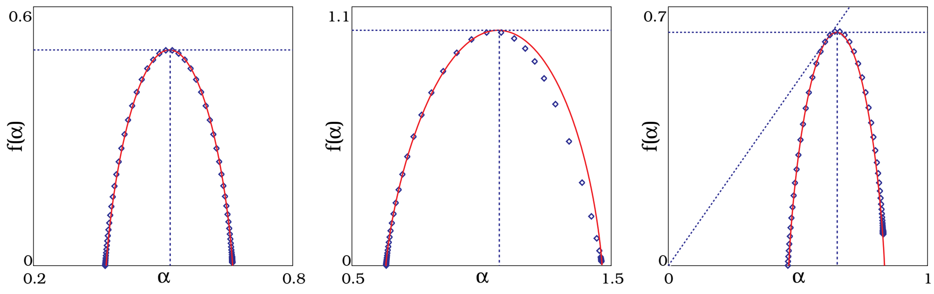

Figure 6. Legendre tranform $f(\alpha)$ of the function $\tau(q)$ computed by the discretization of the correlation integral $W_C^{(n)}(m, q)$ (dots) and from dynamical partitions (continuous line) for the measures generated by the maps (b), (c) and (d) (from left to right) on the unit interval.

a consequence the whole $f(\alpha)$ spectrum of local exponents appears to be significant for a generic image. Applications to the analysis of biomedical images are under consideration. In particular, the $f(\alpha)$ spectrum seems to be capable of discriminating between normal and osteoporotic bone radiographs.

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We would like to thank S Vaienti for useful discussions.

Appendix

The computational complexity of the algorithms proposed to evaluate $W^{(n)}(m; q)$, defined by (4.6) can be evaluated as follows. At step m (recall that $m = n$ is the order of the partition corresponding to the pixel scale whereas $m = 0$ corresponds to the unit d-dimensional cube) the number of sums needed to evaluate $g_i^{(m)}$ from the g_j is $2^{(n-m)d}$, where i runs from 1 to 2^{md}. As a consequence, the evaluation of the $W^{(n)}(m; q)$ for any m is the same 2^{nd} and since m varies between 1 and n the computational complexity is $n2^{nd}$ or letting $M = 2^{nd}$ be the number of pixels it can be written as

$$C_M = \frac{M}{d} \log_2 M.$$

The optimal algorithm is defined by the iteration where $g_i^{(m)}$ is computed from $g_j^{(m+1)}$ where m runs from $n - 1$ to 0. In this case there are 2^d sums to compute each $g_i^{(m)}$ and consequently $2^{(m+1)d}$ to compute all of them. In this case the computational complexity of $W^{(n)}(m; q)$ is

$$C_M = 2^{2d} \frac{(M-1)}{2^d - 1}.$$

We remark that the result is linear in M and that for $d = 1$ we have $C_M = 4(M - 1)$, whereas for $d = 2$ we have $C_M = \frac{8}{3}(M - 1)$. In section 4 we have used $M = N^2$.

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